

# Hamiltonian Hydrodynamics and Irrotational Binary Inspiral

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Gravitational waves from neutron-star and black-hole binaries carry valuable information on their physical properties and probe physics inaccessible to the laboratory. Although development of black-hole gravitational-wave templates in the past decade has been revolutionary, the corresponding work for double neutron-star systems has lagged. Neutron stars can be well-modelled as simple barotropic fluids during the part of binary inspiral most relevant to gravitational wave astronomy, but the crucial geometric and mathematical consequences of this simplification have remained computationally unexploited. In particular, Carter and Lichnerowicz have described barotropic fluid motion via classical variational principles as conformally geodesic. Moreover, Kelvin's circulation theorem implies that initially irrotational flows remain irrotational. Applied to numerical relativity, these concepts lead to novel Hamiltonian or Hamilton-Jacobi schemes for evolving relativistic fluid flows. Hamiltonian methods can conserve not only flux, but also circulation and symplecticity, and moreover do not require addition of an artificial atmosphere typically required by standard conservative methods. These properties can allow production of high-precision gravitational waveforms at low computational cost. This canonical hydrodynamics approach is applicable to a wide class of problems involving theoretical or computational fluid dynamics.

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**Introduction.**—A wide variety of compact stellar objects where general relativistic effects are important is currently known. Black holes and neutron stars are involved in many astrophysical phenomena, including binary mergers and gamma ray bursts, which have observable imprints in the electromagnetic and gravitational wave spectrum. Many of these phenomena can be modelled by means of general relativistic hydrodynamics. In particular, flows describing cosmological fluid expansion [1], certain types of accretion [2–6], binary neutron star [7–13] or black hole-neutron star [14–16] inspiral and other phenomena, can be well-modelled as irrotational.

With gravitational-wave astronomy about to become a reality, and given that inspiral signal detection and parameter estimation typically requires prior theoretical knowledge of the waveforms, great effort has been made towards source modelling and accurate waveform template construction. Although development of black-hole gravitational wave templates in the past decade has been revolutionary, the corresponding work for neutron-star systems has lagged in accuracy due to the presence of matter [17–22].

Barotropic flows accurately model binary neutron stars in their inspiral phase [23, 24]. Synge [25] and Lichnerowicz [26] have shown that relativistic barotropic flows may be described via classical variational principles as conformally geodesic. Carter [27] used a non-affinely parametrized action to construct a super-Hamiltonian and to elegantly derive covariant 4-dimensional hydrodynamic conservation laws.

In an effort towards ‘clean’ gravitational waveforms, this paper outlines a canonical hydrodynamics approach that provides insight, technical simplification and gain in efficiency and accuracy to problems involving binary inspiral. To this end, we adopt Carter’s framework but introduce a 3-dimensional *constrained* Hamiltonian based on an affinely parametrized action. We obtain variational principles in a

covariant 3+1 form valid for both Newtonian gravity and general relativity. Moreover, we exploit the implications of Kelvin’s theorem for relativistic irrotational hydrodynamics and construct a strongly hyperbolic evolution scheme with novel properties, applicable to binary neutron star inspiral and other problems. Notably, the constrained Hamiltonian approach is strictly flux-conservative and naturally eliminates the need for an artificial atmosphere, typically required by conservative methods (see also [28] for a level-set approach). Additionally, this approach has promising applications in theoretical [29, 30] and computational [31–33] fluid dynamics, in a wide variety of Newtonian and relativistic contexts.

Below we outline our constrained Hamiltonian formulation and its features; we relegate full details and results to a forthcoming paper. Generalization to non-irrotational or non-barotropic flows is also deferred to future work. Spacetime indices are Greek and spatial indices Latin. We set  $G = c = 1$  and use  $\nabla_a$  or  $\partial_a$  to denote the (Eulerian) covariant or partial derivative compatible with a curved or flat metric respectively, and  $\partial/\partial x^\alpha$  to denote the (Lagrangian) partial derivative of a function  $f(x, v)$  with respect to  $x^\alpha$  for fixed  $v^\beta$ .

**Barotropic thermodynamics.**—Consider a perfect fluid with proper energy density  $\epsilon$  and pressure  $p$ . Let us assume that the fluid is a *simple barotropic fluid*, that is, all thermodynamic quantities depend only on rest-mass density  $\rho$  and the fluid is ‘cold’ (zero temperature) or homentropic. For barotropic fluids, the specific enthalpy  $h$  is equal to the chemical potential and satisfies the Gibbs-Duhem relation [23, 24, 34]

$$h(\rho) := \frac{\epsilon + p}{\rho} = 1 + \int_0^\rho \frac{dp}{\rho} = 1 + \eta \quad (1)$$

where  $\eta$  is the kinetic specific enthalpy, satisfying  $\eta \ll 1$  in the Newtonian limit. The relation  $\rho = \rho(h)$  is the equation of state (EOS) of the fluid.

*Euler-Lagrange hydrodynamics.*—To set the stage, we review Euler-Lagrange dynamics in covariant language as applied to fluid theory in Newtonian gravity or 3+1 general relativity (the relativistic four-dimensional formulation is outlined in [27] and its generalization to magnetohydrodynamics is given in [35]). The results derived in this paper will apply to any motion in which the flow lines obey a Lagrangian variation principle. That is, for any particular flow configuration, there exists a Lagrangian function  $L(t, x, v)$  of the spacetime coordinates  $x^\alpha = \{t, x^a\}$  and *canonical 3-velocity*  $v^a = dx^a/dt$  of a fluid element measured in local coordinates. Consider a fluid element of unit mass moving along a streamline under the influence of pressure and gravitational forces. We assert that, in both nonrelativistic and relativistic contexts, and for both self-gravitating or test fluids, the motion of a fluid element can be obtained from an action of the form

$$S = \int_{t_1}^{t_2} L(t, x, v) dt \quad (2)$$

Minimizing the action yields the Euler-Lagrange equation of motion:

$$\frac{dp_a}{dt} - \frac{\partial L}{\partial x^a} = (\partial_t + \mathcal{L}_v)p_a - \nabla_a L = 0 \quad (3)$$

where  $p_a = \partial L(t, x, v)/\partial v^a$  is the *canonical momentum* of the fluid element conjugate to  $x^a$ ,  $\mathcal{L}_v$  is the Lie derivative along  $v^a$  and  $\nabla_a L := \partial L(t, x, v)/\partial x^a + p_b \partial v^b/\partial x^a$ . As emphasized by Carter [27], the second, covariant version of Eq. (3) is the form appropriate in a fluid-theory context.

For a barotropic fluid, Eq. (1) implies that the pressure force arises from a potential. Then, the nonrelativistic Lagrangian

$$L(t, x, v) = \frac{1}{2}\gamma_{ab}(x)v^a v^b - \Phi(t, x) - \eta(t, x) \quad (4)$$

(where  $\gamma_{ab}$  is the Euclidian 3-metric and  $\Phi$  is the Newtonian potential), implies

$$p_a = \frac{\partial L}{\partial v^a} = \gamma_{ab}v^b = v_a \quad (5)$$

and, when substituted into Eq. (3), yields the nonrelativistic Euler equation in Lagrangian form:

$$(\partial_t + \mathcal{L}_v)v_a = \nabla_a(\frac{1}{2}v^2 - \Phi - \eta) \quad (6)$$

where  $v^2 = v_b v^b$ . Barotropic fluid motion may thus be described as motion in an effective potential  $\Phi + \eta$ . In the pressureless ('dust') limit,  $\eta$  vanishes and the motion reduces to that of a particle in a Newtonian potential  $\Phi$ .

An analogous result holds in general relativity: barotropic fluid streamlines are *geodesics* of a Riemannian manifold with metric  $h^2 g_{\alpha\beta}$  [26]. These geodesics minimize the arc length  $S = -\int_{\tau_1}^{\tau_2} h(x) \sqrt{-g_{\alpha\beta}(x)u^\alpha u^\beta} d\tau$ , where  $g_{\alpha\beta}$  is the spacetime metric,  $u^\alpha = dx^\alpha/d\tau = u^i dx^i/dt$  is the fluid 4-velocity and  $\tau$  is the proper time of an observer comoving with the fluid [34]. With the standard 3 + 1 decomposition, the spacetime  $\mathcal{M} = \mathbb{R} \times \Sigma$  is foliated by a family of spacelike surfaces  $\Sigma_t$  and,

in a chart  $\{t, x^i\}$ , its metric takes the form  $d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu = \alpha^2 dt^2 - \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$ , where  $\alpha$  is the lapse,  $\beta^a$  is the shift vector and  $\gamma_{ab}$  is the spatial metric. Substituting the 3+1 metric into the action  $S$  and using the coordinate time  $t$  as affine parameter leads to an action of the form (2) with the relativistic Lagrangian given by

$$L(t, x, v) = -\alpha(t, x)h(t, x) \sqrt{1 - \gamma_{ab}(t, x)v^a v^b} = -h/u^t, \quad (7)$$

where  $v^a = \alpha^{-1}(v^a + \beta^a)$  is the fluid 3-velocity measured by *normal observers*. The canonical 3-momentum is given by

$$p_a = \frac{\partial L}{\partial v^a} = h \frac{v_a}{\sqrt{1 - v^2}} = hu_a, \quad (8)$$

where  $v_a = \gamma_{ab}v^b$  and  $v^2 = v_b v^b$ . Substituting Eqs. (7) and (8) into Eq. (3) yields the relativistic Euler equation in Lagrangian 3+1 form:

$$(\partial_t + \mathcal{L}_v)(hu_a) = -\nabla_a(h/u^t). \quad (9)$$

This equation could have been obtained directly by 3+1 decomposing the relativistic Euler equation in four-dimensional Lagrangian form,  $\mathcal{L}_u(hu_a) = -\nabla_a h$  [23, 24, 27], but its derivation from a variational principle is essential for what follows. In the pressureless limit,  $h = 1$ , the motion reduces to a geodesic of  $\mathcal{M}$ . Eqs. (6) and (9) are suitable for numerical evolution in Lagrangian coordinates, such as smoothed-particle hydrodynamics. The canonical approach outlined below is suited to methods based on either Eulerian or Lagrangian coordinates.

*Hamiltonian hydrodynamics.*—Using the covariant Euler-Lagrange equation (3) and the Cartan identity  $\mathcal{L}_v p_a = v^b(\nabla_b p_a - \nabla_a p_b) + \nabla_a(v^b p_b)$ , one obtains the covariant Hamilton equation

$$\frac{dp_a}{dt} + \frac{\partial H}{\partial x^a} = \partial_t p_a + v^b(\nabla_b p_a - \nabla_a p_b) + \nabla_a H = 0 \quad (10)$$

where

$$H(t, x, p) = v^a p_a - L(t, x, v) \quad (11)$$

is the Hamiltonian of a fluid element. Note that, like Eq. (3), Eq. (10) is valid in Newtonian *and* relativistic contexts.

For nonrelativistic barotropic flows, Eqs. (4), (5) and (11) yield the Hamiltonian

$$H(t, x, p) = \frac{1}{2}\gamma^{ab}(x)p_a p_b + \Phi(t, x) + \eta(t, x) \quad (12)$$

and Eq. (10) yields the nonrelativistic Euler equation in canonical form, also known as the Crocco equation:

$$\partial_t v_a + v^b(\nabla_b v_a - \nabla_a v_b) + \nabla_a(\frac{1}{2}v^2 + \Phi + \eta) = 0. \quad (13)$$

Multiplying this equation by the density  $\rho$  and using the Gibbs-Duhem relation (1) and the nonrelativistic continuity equation

$$\partial_t \rho + \nabla_a(\rho v^a) = 0 \quad (14)$$

leads to a flux-conservative form of the Euler equation:

$$\partial_t(\rho v_a) + \nabla_b T_a^b = -\rho \nabla_b \Phi. \quad (15)$$

where  $T_a^b = \rho v_a v^b + p \gamma_a^b$  is the fluid stress tensor.

For relativistic barotropic flows, Eqs. (7), (8) and (11) yield the constrained Hamiltonian

$$H(t, x, p) = -p_a \beta^a(t, x) + \alpha(t, x) \sqrt{h(t, x)^2 + \gamma^{ab}(t, x) p_a p_b} = -h u_t \quad (16)$$

and Eq. (10) yields the relativistic Euler equation in 3+1 canonical form

$$\partial_t(h u_a) + v^b [\nabla_b(h u_a) - \nabla_a(h u_b)] - \nabla_a(h u_t) = 0. \quad (17)$$

This equation could have been obtained by 3+1 decomposing the Euler equation in Carter-Lichnerowicz form, written as  $u^\beta [\nabla_\beta(h u_\alpha) - \nabla_\alpha(h u_\beta)] = 0$  in four dimensions [23, 25–27]. The Hamiltonian (16) amounts to the energy of a fluid element measured in local coordinates and could have alternatively been obtained by solving the constraint  $g^{\alpha\beta} u_\alpha u_\beta = -1$  for  $u_t$ . In the pressureless limit,  $h = 1$ , Eq. (16) reduces to the constrained Hamiltonian of a particle of unit mass moving on a spacetime geodesic [36] and Eq. (17) describes a congruence of such geodesics.

Multiplying Eq. (17) by the density  $\rho$  and using the Gibbs-Duhem relation (1) and the relativistic continuity equation

$$\nabla_\alpha(\rho u^\alpha) = \frac{1}{\sqrt{-g}} \partial_\alpha(\sqrt{-g} \rho u^\alpha) = 0 \quad (18)$$

(where  $g = \det(g_{\mu\nu})$ ) implies that the divergence of the fluid energy-momentum tensor  $T_\alpha^\beta = \rho h u_\alpha u^\beta + p g_\alpha^\beta$  vanishes:

$$\nabla_\beta T_\alpha^\beta = \frac{1}{\sqrt{-g}} \partial_\beta(\sqrt{-g} T_\alpha^\beta) - \Gamma_{\alpha\beta}^\gamma T_\gamma^\beta = 0. \quad (19)$$

The above flux-conservative form of the Euler equation is typically used in numerical simulation via shock-capturing methods. However, the canonical form (10) carries unique advantages, especially in the irrotational case discussed below.

*Conservation of circulation.*—The canonical vorticity 2-form,  $\omega_{ab} := \nabla_a p_b - \nabla_b p_a$ , satisfies an evolution equation,  $(\partial_t + \mathcal{L}_v)\omega_{ab} = 0$ , obtained from the exterior derivative of Eq. (3). The integral form of this equation constitutes Kelvin's circulation theorem: the circulation along a fluid ring  $C_t = \partial S_t$  dragged along by the flow is conserved:

$$\frac{d}{dt} \oint_{C_t} p_a dx^a = \frac{d}{dt} \int_{S_t} \omega_{ab} d\Sigma^{ab} = \int_{S_0} (\partial_t + \mathcal{L}_v) \omega_{ab} d\Sigma^{ab} = 0 \quad (20)$$

where the first equality follows from the Stokes theorem and represents the flux of vorticity through the surface  $S_t = \Psi_t S_0$ , where  $\Psi_t$  is the family of diffeomorphisms generated by fluid velocity  $v^a$ .

From a general variation of the action (2), it is possible to show [37, 38] that the integral

$$I = \oint_C (p_a dx^a - H dt), \quad (21)$$

calculated along an arbitrary closed contour  $C = \partial S$  lying on the hypersurface  $S$  (to which the fluid motion is restricted) of the extended phase space  $(x^a, p_a, t)$ , is invariant under an arbitrary displacement or deformation of the contour along any tube of fluid streamlines (or particle trajectories in the pressureless limit). A dynamical system admits an invariant  $I$ , known as the *Poincaré-Cartan integral invariant*, iff it is Hamiltonian. If we consider curves  $C_t$  lying in planes of constant  $t$  in phase space, then  $dt = 0$  along such curves and  $I$  reduces to the conserved circulation integral in (20). In four dimensional general relativity, one typically evaluates the integral (21) along a fluid ring  $C_\tau$  of constant proper time  $\tau$  and writes Kelvin's theorem in the form  $\frac{d}{d\tau} \oint_{C_\tau} p_a dx^a = 0$ ; this conservation law can be derived directly from the relativistic Euler equation [24]. We stress, nevertheless, the fact that Eqs. (20), (21) are valid as written in both Newtonian gravity and 3+1 general relativity.

The most interesting feature of Kelvin's theorem is that, since its derivation did not depend on the metric, it is exact in time-dependent spacetimes, with gravitational waves carrying energy and angular momentum away from a system. In particular, *oscillating stars and radiating binaries, if modeled as barotropic fluids with no viscosity or dissipation other than gravitational radiation, exactly conserve circulation* [24]. An important corollary of Kelvin's theorem is that, if circulation is zero initially, it must remain zero subsequently. That is, *flows initially irrotational remain irrotational*. Apart from an application to incompressible Newtonian binaries [39], this concept has remained unexploited in simulations of binary inspiral, despite the fact that numerical relativity simulations typically begin with irrotational neutron-star initial data [7–16, 23], as spin is usually negligible in this regime. The implications of this corollary for relativistic fluid dynamics are explored below.

*Irrotational Hamilton-Jacobi hydrodynamics.*—A flow is called irrotational if the vorticity 2-form  $\omega_{ab}$  vanishes

$$\nabla_a p_b - \nabla_b p_a = 0 \quad (22)$$

or, by virtue of the Poincaré lemma (for simply connected manifolds), if the canonical momentum is the gradient of a velocity potential:

$$p_a = \nabla_a S(t, x) \quad (23)$$

For irrotational flows, the Hamilton equation (10) simplifies to the strictly flux-conservative canonical equation

$$\partial_t p_a + \nabla_a H(t, x, p) = 0 \quad (24)$$

Substituting Eq. (23) into (24) gives the first integral

$$\partial_t S(t, x) + H(t, x, \nabla S) = 0 \quad (25)$$

which has the form of a Hamilton-Jacobi equation. (The integration constant  $c(t)$  is eliminated by adding  $\int^t c(t') dt'$  to  $S$  without altering  $p_a$ .)  $H$  and  $p_a$  are given by Eqs. (12), (5) for Newtonian gravity or Eqs. (16), (8) for 3+1 general relativity.

The above corollary to Kelvin’s theorem suggests that irrotational initial data may be evolved by solving either the Hamilton-Jacobi equation (25) or its gradient, the Hamilton equation (24). This is *equivalent* to solving the Euler equation – there is no approximation involved – as long as the initial data is irrotational. (In fact, even for a non-barotropic EOS, the Euler equation may be used to show that initially irrotational flows are also initially barotropic, i.e. homentropic or zero temperature, and remain so subsequently).

For the Hamiltonian functions given above, Eq. (24) can be considered a generalization of the Burgers equation. In the absence of pressure and gravitational forces, by virtue of Eqs. (12) and (5), Eq. (24) reduces to the nonrelativistic Burgers equation,  $\partial_t v_a + \partial_a(\frac{1}{2}v^2) = 0$ . In Minkowski space, by virtue of Eqs. (16) and (8), Eq. (24) similarly reduces to a special-relativistic Burgers equation,  $\partial_t(v_a/\sqrt{1-v^2}) + \partial_a(\sqrt{1+v^2}) = 0$ , which reduces to the nonrelativistic equation for  $v \ll 1$ . LeFloch et al. [40, 41] provide a non-covariant derivation of this equation for Minkowski and Schwarzschild spacetimes, based on algebraic manipulation of the Euler and continuity equations on particular charts rather than covariant variational principles; numerical evolutions of these equations in 1+1 dimensions were successful, even in the presence of shocks. However, the fact that such equations amount to Hamilton or Hamilton-Jacobi equations, that can be obtained from constrained particle-like variational principles and written in covariant 3+1 form for *any* spacetime, remains unnoticed. The covariant approach outlined above motivates the use of Eqs. (8), (16) and (24) or (25) for irrotational hydrodynamics in a variety of physical contexts.

Several methods (cf. [42–47] and references therein) exist for solving Hamilton-Jacobi equations numerically. A well-known mathematical problem encountered with such equations is non-uniqueness of solutions, but unique ‘viscosity solutions’ may be obtained in the limit of small viscosity [48]. Eq. (25) provides the possibility of applying such well-established methods in the context of Newtonian or relativistic fluid dynamics. Although this equation has the advantage of being scalar, there are certain advantages to using its flux-conservative canonical form (24) for computational purposes. In the latter approach, one may make use of existing flux-conservative scheme, abundantly implemented in numerical relativity, without artificial viscosity, but must check that the constraint (22) is satisfied; such violations also appear in the standard approach and may be eliminated via relaxation techniques [42]. The canonical equation (24) is coupled, via a barotropic equation of state  $\rho = \rho(h)$ , to the continuity equation. In general relativity, the latter is given by Eq. (18) and can be decomposed as

$$\partial_t \rho_\star + \partial_a(\rho_\star v^a) = 0 \quad (26)$$

where  $\rho_\star := \sqrt{-g}\rho u^t = \alpha \sqrt{\gamma}\rho u^t$  and  $\gamma = \det(\gamma_{ij})$ . Then, the system of Eqs. (24) and (26) can be written as

$$\partial_t \mathbf{U} + \partial_k \mathbf{F}^k = 0 \quad (27)$$

where the components of the conservative variable vector  $\mathbf{U}$  and flux vectors  $\mathbf{F}^k$  are given by

$$\mathbf{U} = \begin{pmatrix} \rho_\star \\ p_i \end{pmatrix}, \quad \mathbf{F}^k = \begin{pmatrix} \rho_\star v^k \\ \delta_i^k H \end{pmatrix}, \quad k = 1, 2, 3 \quad (28)$$

and  $p_i, H$  are given by Eqs. (8), (16). In the Newtonian limit, one sets  $\rho_\star := \sqrt{\gamma}\rho$  and uses Eqs. (5), (12) instead.

Eq. (27) can be evolved together with the spacetime metric [49–51] and is our main result. Notably, this evolution system is source-free, and thus *strictly* flux-conservative, with no further assumptions such as Killing symmetries. Moreover, for finite sound speed  $c_s = \sqrt{dp/d\epsilon} = \sqrt{d \ln h / d \ln \rho}$ , the system is *strongly hyperbolic* and thus has a well-posed initial value problem: a lengthy but straightforward characteristic analysis shows that the system possesses a complete basis of four eigenvectors, with eigenvalues  $\lambda_{1,2}^k = 0$  (double) and  $\lambda_{3,4}^k = \alpha(1 - v^2 c_s^2)^{-1} \{v^k(1 - c_s^2) \pm c_s(1 - v^2)^{1/2}[(1 - v^2 c_s^2)\gamma^{kk} - (1 - c_s^2)(v^k)^2]^{1/2}\} - \beta^k$ . The latter pair of ‘acoustic’ eigenvalues is identical to those of the Valencia formulation, while the former pair is different [52]. When numerically evolving Eq. (27), one needs to construct the fluxes  $\mathbf{F}^k$  given the conserved variables  $\mathbf{U}$  at each time step. To do so, one needs to recover the primitive variables  $\{h, u_i\}$  given  $\mathbf{U}$ , by first solving for  $h$  the algebraic equation

$$\rho(h) = \frac{\rho_\star h}{\sqrt{\gamma} \sqrt{\gamma^{ij} p_i p_j + h^2}} \quad (29)$$

for fixed  $\rho_\star, p_i$  and  $\gamma_{ij}$ . This equation is obtained by substituting the relation  $u^t = \alpha^{-1} \sqrt{\gamma^{ij} u_i u_j + 1}$  into the definition of  $\rho_\star$  and using Eq. (8). A novel feature of Eq. (27) is that the recovery of  $u_i$  is performed by dividing  $p_i$  by the specific enthalpy  $h$  which becomes unity on the surface, rather than dividing  $\rho_\star u_i$  by the density  $\rho$  which vanishes there. Thus, unlike the standard approach, *no artificial atmosphere is required for recovery of primitive from conservative variables*.

**Conclusions.**—Although the Carter-Licherowicz approach [26, 27] has been used to obtain first integrals for constructing initial data for compact binaries in the presence of Killing symmetries [7–16, 23], it has never been adopted to fluid flow evolution. Moreover, since irrotationality is independent of helical symmetry, this simplification applies not only to circular but also inspiralling or eccentric nonspinning binaries, but has yet to be exploited in hydrodynamic simulations. This paper provides the steps towards these goals. Numerical tests of the irrotational hydrodynamics system (27) have been performed successfully; details and results from simulation of binary neutron star inspiral will be provided in a future paper.

Avoiding an artificial atmosphere does not only increase accuracy (as systematic errors related to the atmosphere are eliminated) but also increases efficiency (as numerical operations for hydrodynamics outside the star are avoided). As mentioned earlier, unlike the energy-momentum conservation laws (19), the irrotational conservation laws (27) are source-free and represent *strict* conservation. This feature simplifies implementation and increases precision as it avoids numerical



differentiation of the metric. A caveat is that the Hamiltonian is nondifferentiable at the star surface, so care must be taken in performing numerical differentiation at that location to retain accuracy, as detailed elsewhere. Additional accuracy can be gained by using *symplectic integration* schemes for time evolution, that preserve Hamiltonian structure and circulation.

Finally, although the above approach focused on irrotational flows, it is feasible to accommodate non-irrotational or even non-barotropic flows in the formulation while retaining most of its merits. Such developments are expected to be of interest in theoretical and computational fluid dynamics, in Newtonian and relativistic contexts, and motivate future work.

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